Math 010 Exam 2
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. Let the transformation T be defined by $T(x_1, x_2, x_3) = (x_1 + 5x_2,\ 3x_1 - x_2 + 2x_3)$.
a. (3 pts) Find the standard matrix $[T]$.

$$[T] = \begin{bmatrix} 1 & 5 & 0 \\ 3 & -1 & 2 \end{bmatrix}$$

b. (1 pt) State the domain and codomain of T.

Domain: R^3

Codomain: R^2

c. (2 pts) Use the matrix you found in part (a) to compute $T(2, 1, -4)$.

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+5 \\ 6-1-8 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

2. (6 pts) Use the linearity conditions to determine whether $T: R^2 \to R^3$ defined by $T(x_1, x_2) = (x_1 - x_2,\ 0,\ 2x_1 + 1)$ is a linear transformation. Be sure to justify your answer.

ii) $T(k\vec{x}) = T(kx_1, kx_2)$

$= (kx_1 - kx_2,\ 0,\ 2kx_1 \underline{+1})$

$kT(\vec{x}) = (kx_1 - kx_2,\ 0,\ 2kx_1 \underline{+k})$

$\neq T(k\vec{x})$

This is not a linear transformation.

3. a. (6 pts) Find the standard matrix for the stated composition in R^2.
Reflection about the line $y = x$, followed by a counterclockwise rotation by $180°$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad R_{180°} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$R_{180°} A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

b. (2 pts) Compute the image of $(-2, 5)$ under this transformation.

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

4. (8 pts) Find all values of λ for which $\det(A) = 0$, where

$$A = \begin{bmatrix} \lambda - 4 & 1 & 0 \\ 0 & \lambda - 2 & 0 \\ 3 & 1 & \lambda - 1 \end{bmatrix}$$

$$(\lambda - 1)\begin{vmatrix} \lambda - 4 & 1 \\ 0 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 1)\left[(\lambda - 4)(\lambda - 2) - 0\right] = 0$$

$$\lambda = 1, 2, 4$$

5. (10 pts) Find the polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ that passes through the three points $(1,3)$, $(2,5)$, and $(3,9)$ by row reducing an appropriate augmented matrix. (*Note: You don't need to take it all the way to RREF or even row echelon form. Row reduce until you can easily obtain the answer.*)

$(1,3)$: $a_0 + a_1 + a_2 = 3$

$(2,5)$: $a_0 + 2a_1 + 4a_2 = 5$

$(3,9)$: $a_0 + 3a_1 + 9a_2 = 9$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 5 \\ 1 & 3 & 9 & 9 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\begin{array}{cccc} 1 & 2 & 4 & 5 \\ -1 & -1 & -1 & -3 \\ \hline 0 & 1 & 3 & 2 \end{array}$$

$R_3 \to R_3 - R_1$

$$\begin{array}{cccc} 1 & 3 & 9 & 9 \\ -1 & -1 & -1 & -3 \\ \hline 0 & 2 & 8 & 6 \end{array}$$

$\hookrightarrow 0 \ 1 \ 4 \ 3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 4 & 3 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\begin{array}{cccc} 0 & 1 & 4 & 3 \\ 0 & -1 & -3 & -2 \\ \hline 0 & 0 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3 \Rightarrow a_2 = 1$, then $R_2 \Rightarrow a_1 + 3 = 2$

$a_1 = -1$

$R_1 \Rightarrow a_0 - 1 + 1 = 3 \Rightarrow a_0 = 3$

$$P(x) = 3 - x + x^2$$

6. (2 pts each) Suppose A is a 3×3 matrix with $\det(A) = 3$. Compute each of the following.

a. $\det(2A)$

$$2^3 \det(A) = 24$$

b. $\det(A^T)$ $= \det(A) = 3$

c. $\det(A^2)$ $= [\det(A)]^2 = 3^2 = 9$

d. $\det((3A)^{-1})$ $= \dfrac{1}{\det(3A)} = \dfrac{1}{3^3 \cdot 3} = \dfrac{1}{81}$

7. (6 pts) Find a and b such that $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$, where $\mathbf{u} = (2, 1, 3)$, $\mathbf{v} = (1, 2, -1)$, and $\mathbf{w} = (5, 4, 4)$ or show that no such scalars exist.

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 2 & 4 \\ 3 & -1 & 4 \end{array}\right]$$

$R_2 \Rightarrow R_1 - 2R_2$

$$\begin{array}{ccc} 2 & 1 & 5 \\ -2 & -4 & -8 \\ \hline 0 & -3 & -3 \end{array}$$

$\hookrightarrow 0 \quad 1 \quad 1$

$R_3 \Rightarrow R_3 - 3R_2$

$$\begin{array}{ccc} 3 & -1 & 4 \\ -3 & -6 & -12 \\ \hline 0 & -7 & -8 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & -7 & -8 \end{array}\right]$$

$R_3 \Rightarrow R_3 + 7R_2$

$$\begin{array}{ccc} 0 & -7 & -8 \\ 0 & 7 & 7 \\ \hline 0 & 0 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

$R_3 \Rightarrow 0 = -1$, which is a contradiction.

No such a & b exist.

8. (2 pts each) Let $\mathbf{u} = (2, -1, 2)$ and $\mathbf{v} = (1, 2, -1)$. Compute each of the following:
a. $\|3\mathbf{u} + 2\mathbf{v}\|$

$$3\vec{u} + 2\vec{v} = (6, -3, 6) + (2, 4, -2)$$
$$= (8, 1, 4)$$
$$\|3\vec{u} + 2\vec{v}\| = \sqrt{64 + 1 + 16} = 9$$

b. $\mathbf{u} \cdot \mathbf{v}$

$$2 \cdot 1 - 1(2) + 2(-1) = -2$$

c. Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? Please justify your answer.

No because $\vec{u} \cdot \vec{v} \neq 0$.

d. Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{-2}{3\sqrt{6}}$$

9. (4 pts) Let $\mathbf{u} = (2, 5, 4)$ and $\mathbf{a} = (2, 2, 1)$. Find the following:

a. The vector projection of $\mathbf{u}$ onto $\mathbf{a}$, $\text{proj}_{\mathbf{a}}\mathbf{u}$.

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} = \frac{4+10+4}{9} \vec{a}$$

$$= 2\vec{a} = (4, 4, 2) = \vec{w}_1$$

b. (3 pts) The component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ and verify that it is orthogonal to $\mathbf{a}$.

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = (2, 5, 4) - (4, 4, 2)$$

$$= (-2, 1, 2)$$

$$\vec{w}_2 \cdot \vec{a} = (-2, 1, 2) \cdot (4, 4, 2) = 8 - 4 - 4 = 0$$

$$\vec{w}_2 \perp \vec{a} \checkmark$$

c. (3 pts) Confirm that the results from (a) and (b) add up to $\mathbf{u}$.

$$(4, 4, 2) + (-2, 1, 2) = (2, 5, 4) = \vec{u} \checkmark$$

10. (8 pts) Prove the distributive property of scalar multiplication for vectors: $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

Let $\vec{u} = (u_1, u_2, \dots, u_n)$, $\vec{v} = (v_1, v_2, \dots, v_n)$ and $k \in \mathbb{R}$.

$k(\vec{u}+\vec{v}) = k(u_1+v_1, u_2+v_2, \dots, u_n+v_n)$ by the definition of vector addition

$= (k(u_1+v_1), k(u_2+v_2), \dots, k(u_n+v_n))$ by the def. of scalar mult.

$= (ku_1+kv_1, ku_2+kv_2, \dots, ku_n+kv_n)$ by the distributive property for real #s.

$= (ku_1, ku_2, \dots, ku_n) + (kv_1, kv_2, \dots, kv_n)$ by the def. of vector addition

$= k\vec{u} + k\vec{v}$ ✓

11. (10 pts) Let $T_A(\mathbf{x}) = A\mathbf{x}$ and $T_B(\mathbf{x}) = B\mathbf{x}$ be matrix transformations. Prove that $T_A \circ T_B$ is invertible if and only if matrices A and B are both invertible.

$(T_A \circ T_B)(\vec{x}) = T_A(B\vec{x}) = A(B\vec{x}) = (AB)\vec{x} = T_{AB}\vec{x}$

$T_A \circ T_B$ is invertible $\Leftrightarrow$ T_{AB} is invertible

But $(T_{AB})^{-1} = T_{(AB)^{-1}}$, so $(T_{AB})^{-1}$ exists

$\Leftrightarrow (AB)^{-1}$ exists $\Leftrightarrow B^{-1}A^{-1}$ exists

$\Leftrightarrow A$ & B are invertible.